The Algebra of Logical Atomism

Peter Fritz and Andrew Bacon

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Abstract

Central to certain versions of logical atomism is the claim that every proposition is a truth-functional combination of elementary propositions. Assuming that propositions form a Boolean algebra, we consider five natural formal regimentations of this informal claim, and show that they are equivalent. For a number of reasons, such as the need to accommodate quantifiers, logical atomists might consider only complete Boolean algebras, and take into account infinite truth-functional combinations. We show that in such a variant setting, the five regimentations come apart, and explore how they relate to each other. We also discuss how they relate to the claim that propositions form a double powerset algebra, which has been proposed by a number of authors as a way of capturing the central logical atomist idea.

Russell (1956 [1918], 1924/1956) and Wittgenstein (1921) defended versions of *logical atomism*. It is not a straightforward matter to explain exactly what logical atomism amounts to, in part because Russell's and Wittgenstein's views changed over time, and differ in various ways from each other. One idea which is central to many versions of logical atomism is the claim that there is a special class of *elementary* propositions (sometimes also called *atomic*) from which all others can be obtained. An especially strong and intriguing form of this claim holds that every proposition can be obtained from elementary propositions using truth-functional operations alone. That is:

(LA) Every proposition is a truth-functional combination of elementary propositions.

Wittgenstein (1921) articulated this idea explicitly in various passages (see 4.51, 5, 5.3, 5.32, 6.001). Russell may have held a version of it as well at some point, but his most explicit developments of logical atomism in (Russell, 1956 [1918], 1924/1956) arguably contradict this strong principle LA.

In the following, we consider a range of ways of formally regimenting LA and some related principles. The aim of this paper is systematic, rather than historical. We therefore focus on exploring the various formalizations only from a mathematical standpoint, and won't consider to what extent various views are plausibly ascribed to the primary sources on logical atomism. However, various principles we will consider are closely related to claims explicitly formulated by Wittgenstein (1921). For readers interested in such a comparison, we include references to some of the relevant passages. For brevity, these will be noted by writing, e.g., "LPA, 5.3" for "Wittgenstein (1921, 5.3)".

1 Boolean Algebras

We work in an algebraic framework, assuming that propositions form a Boolean algebra (cf. LPA, 4.465, 5.41). That is, propositions are objects and form a set, and the truth-functional operations like negation and conjunction can be understood as functions on this set. Before stating the conditions which Boolean algebras impose on these functions, we should note that the algebraic approach is not our preferred way of regimenting talk of propositions. We prefer to use higher-order quantifiers, in particular so-called propositional quantifiers, i.e., quantifiers binding variables in the position of sentences. For more on this, see Fritz and Jones (2024), Fritz (2024), and Bacon (2024). We follow the algebraic approach here for a number of reasons. First, most existing discussions of principles like LA, such as Skyrms (1993), are couched in algebraic terms. Second, it is interesting to consider alternative approaches to formalization, to understand the import of the choice of framework on the resulting views. Third, the algebraic results here can be used in an instrumental capacity for investigations in terms of higher-order logic, since algebraic structures can be extended in natural ways to models of higher-order logic.

Boolean algebras can be defined in a number of ways. For present purposes, a natural way of doing so requires a Boolean algebra to be a structure $\langle A, -, \wedge \rangle$, where A is a non-empty set, $-$ is a function from A to A, and $\wedge$ is a function from $A \times A$ to A. We adopt the convention of denoting such an algebra using the Gothic letter corresponding to the Roman letter used to denote the underlying set. So, $\langle A, -, \wedge \rangle$ is denoted by $\mathfrak{A}$. When dealing with multiple algebras, we sometimes disambiguate between the operations belonging to different algebras by adding the algebra as an index, writing, e.g., $-_{\mathfrak{A}}$ and $\wedge_{\mathfrak{A}}$, and similarly with various defined operations introduced below. In the present context, we can think of A as the set of propositions. We can correspondingly think of $-$ as negation, which is identified with the function which maps every proposition to its negation. Analogously, we can think of $\wedge$ as conjunction, which maps any pair of propositions to their conjunction.

$\langle A, -, \wedge \rangle$ is a *Boolean algebra* just in case any two truth-conditionally equivalent terms denote the same element. This requires some unpacking. First, from terms "x", "y", "z", ... denoting elements of A, we can construct terms such as "$-x$", "$x \wedge y$", and "$(x \wedge y) \wedge -z$", using the symbols denoting the operations of the algebra and parentheses. Second, these terms can be understood as formulas of propositional logic with primitive logical connectives for negation and conjunction. Third, let two of these terms be *truth-functionally equivalent* just in case they are provably equivalent in classical propositional logic. So, the condition characterizing Boolean algebras given above requires x and $--x$, for example, to denote the same element, since x and $--x$, understood as formulas of propositional logic, are classically equivalent. We can therefore think of the assumption that propositions form a Boolean algebra as capturing the idea that formulas which are provably equivalent in classical propositional logic express the same proposition. This assumption also justifies the omission of other truth-functional operations, such as disjunction, in Boolean algebras. If truth-functionally equivalent terms express the same proposition, then every term truth-functionally equivalent to a disjunction denotes the same proposition as the disjunction. For example, $-(-x \wedge -y)$ can be assumed to be the disjunction of x and y. Consequently, we can define, for any given Boolean algebra, a function $\vee$ mapping any x and y to $-(-x \wedge -y)$, and similarly for the other Boolean connectives such as $\rightarrow$ and $\leftrightarrow$.

This definition of Boolean algebras effectively uses an infinite set of equations. Various of its finite subsets would suffice for the purposes of the definition, but this won't be important in the following; the details can be found in standard textbooks dealing with Boolean algebras such as Davey and Priestley (2002), Givant and Halmos (2009), and Koppelberg (1989).

What we will appeal to at various points is an order $\leq$ which can be associated with every Boolean algebra $\langle A, -, \wedge \rangle$, by letting $x \leq y$ just in case $x \wedge y$ is x, for all x and y. $\leq$ is a *partial order*, which means that it is reflexive on A, transitive, and antisymmetric. We will also appeal to a number of standard notions concerning partial orders. First, to every partial order $\leq$, there corresponds a strict order $<$, where $x < y$ just in case $x \leq y$ and $x \neq y$. Further, in a partial order, x is a *lower bound* of $Y \subseteq A$ if $x \leq y$ for all $y \in Y$; x is an *upper bound* of Y if $y \leq x$ for all $y \in Y$. x is a *greatest lower bound* of Y if x is a lower bound of Y and $z \leq x$ for every lower bound z of Y; x is a *least upper bound* of Y if x is an upper bound of Y and $x \leq z$ for every upper bound z of Y. In a partial order, the greatest lower bound of a set Y, if it exists, is unique, and denoted by $\bigwedge Y$; correspondingly, the least upper bound of a set Y, if it exists, is unique, and denoted by $\bigvee Y$.

Let $\langle A, -, \wedge\rangle$ be a Boolean algebra, and $\leq$ the corresponding order. As the notation suggests, if $Y = \{x, y\}$, then $x \wedge y$ is $\bigwedge Y$, and $x \vee y$ is $\bigvee Y$. This means that every two-element set $\{x, y\}$ has a greatest lower bound and a least upper bound. A partial order satisfying this further constraint is called a *lattice*. Furthermore, a lattice $\leq$ is *bounded* if there are elements $\bot$ and $\top$ such that for all x, $\bot \leq x \leq \top$. ($\bot$ and $\top$ are also called 0 and 1, respectively. Consequently, an element is called *non-zero* if it is distinct from 0, i.e., $\bot$.) If $\leq$ is the order derived from a Boolean algebra, it is bounded: we can take $\bot$ to be $x \wedge -x$, and $\top$ to be $x \vee -x$, for any $x \in A$. Finally, a bounded lattice is *complemented* if every element x has a *complement*, i.e., an element y such that $x \wedge y = \bot$ and $x \vee y = \top$. If $\leq$ is the order derived from a Boolean algebra, it is also complemented: every $x \in A$ has a complement, namely $-x$.

Although this won't be essential in the following – and so readers may skip this paragraph without loss of continuity – it is worth noting that in a certain sense, Boolean algebras can also be defined as certain partial orders. First, the order $\leq$ associated with a given Boolean algebra satisfies one further property, of being *distributive*, which requires that $x \wedge (y \vee z)$ is $(x \wedge y) \vee (x \wedge z)$, for all x, y, and z. So, such an order $\leq$ is always a complemented distributive lattice. In such a lattice, complements are unique. Thus, we can recover $-$ as the function mapping every element to its complement, and $\wedge$ as the function mapping any set $\{x, y\}$ to its greatest lower bounds. Thus, every Boolean algebra $\langle A, -, \wedge\rangle$ uniquely determines a complemented distributive lattice $\leq$, from which we can recover all the components of the original Boolean algebra. Finally, whenever $\leq$ is a complemented distributive lattice, the underlying set A, together with the complementation function $-$ and greatest lower bound function $\wedge$ form a Boolean algebra. Along these lines, Boolean algebras as defined here correspond bijectively to complemented distributive lattices. In this sense, Boolean algebras could also be defined as complemented distributive lattices.

2 Regimenting LA

Assuming that the propositions form a Boolean algebra $\langle A, -, \wedge\rangle$, LA says that the elementary propositions form a set $E \subseteq A$, from which every proposition can be obtained using truth-functional combinations. Formalizing this claim involves two choices: First, what algebraic conditions must elementary propositions satisfy? Second, what does it mean that every proposition is a truth-functional combination of elementary propositions? We will consider two general approaches to answering these questions. Formally, they will lead us to five conditions on a subset E of a Boolean algebra.

2.1 Independent Generation

According to the first approach, the crucial feature of elementary propositions is that they are logically independent (cf. LPA, 4.211, 4.27, 5.134, 6.3751). Informally, logical independence means that for any distinct elementary propositions $e_1, \ldots, e_{n+m}$, it is consistent for $e_1, \ldots, e_n$ to be true, and for $e_{n+1}, \ldots, e_{n+m}$ to be false. That is, the conjunction of $e_1, \ldots, e_n$ with the negations of $e_{n+1}, \ldots, e_{n+m}$ is non-zero (i.e., distinct from 0). Algebraically, this is known as being independent. A useful way of defining this is as follows (in the context of a given Boolean algebra, left implicit):

Definition 2.1. *D is a* finite description based on *E, written $D \lhd_\omega E$, if $D = T \cup \{-f : f \in F\}$ for disjoint finite sets $T, F \subseteq E$. E is* independent *if $\bigwedge D > \bot$ for every $D \lhd_\omega E$.*

It is important to be clear that logical independence is not a way of characterizing elementarity algebraically, since typically, there are multiple independent sets whose union is not independent. In fact, this holds for every non-trivial Boolean algebra, where a Boolean algebra is *trivial* if it has just one element. Up to isomorphism, there is only one trivial Boolean algebra, which we call **1**. For any other Boolean algebra, every element is distinct from its complement. Thus, for any element x, $\{x\}$ and $\{-x\}$ are distinct and independent, but $\{x, -x\}$ is not independent. Consequently, independence is only an algebraic requirement of elementary, not a characterization of it.

Assuming an independent set E of elementary proposition, there is a straightforward way of making sense of the claim that every proposition is a truth-functional combination of elements of E. We can define a sequence of sets $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$, starting from $E_0 = E$, where E_{n+1} contains just the negations and conjunctions of elements of E_n. We can then characterize the propositions which can be obtained from elements of E using truth-functional operations as the union of $E_0, E_1, E_2, \dots$. More formally, we can define:

$$\begin{aligned} E_0 &:= E \\ E_{n+1} &:= \{-x : x \in E_n\} \cup \{y \wedge z : y, z \in E_n\} \\ E_\omega &:= \bigcup_{n \in \mathbb{N}} E_n \end{aligned}$$

To say that every proposition is a truth-functional combination of elements of E is therefore naturally formalized by requiring E_ω to contain every element of A. In algebraic terminology, this requires E to generate $\mathfrak{A}$. It will be useful to introduce an alternative way of stating this requirement, which is equivalent to the one just stated. It uses the notion of a subalgebra. A *subalgebra* of a Boolean algebra $\langle A, -, \wedge \rangle$ is a Boolean algebra based on a subset B of A, such that the operations of the two algebras agree on the elements of B. A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is a *proper subalgebra* of $\mathfrak{A}$ if it is distinct from $\mathfrak{A}$. Subalgebras are closed under intersection: for any set S of subalgebras of a given algebra, the elements contained in every member of S are closed under the operations, and so form a subalgebra themselves. It follows that for a given set $E \subseteq A$, the subalgebras which include E are closed under intersection, and so contain a smallest element. This is the intersection of all subalgebras including E. It can be shown that its underlying set is just E_ω. We can therefore define:

Definition 2.2. *$E \subseteq A$* generates *$\mathfrak{A}$ if the smallest subalgebra of $\mathfrak{A}$ including E is $\mathfrak{A}$ itself.*

The first way of regimenting LA therefore requires the elementary propositions E to be independent and to generate the algebra of all propositions. We introduce an obvious label for this notion:

Definition 2.3. *$E \subseteq A$* independently generates *$\mathfrak{A}$ if E is independent, and generates $\mathfrak{A}$.*

As in the case of independence, a set $E \subseteq A$ which independently generates $\mathfrak{A}$ does not in general do so uniquely. We return to this point in more detail in section 2.3 below.

2.2 Free Generation

The second way of regimenting LA draws on another Tractarian idea, implicit in Wittgenstein's use of truth tables (cf. LPA, 4.442, 5.101). Here the idea that the elementary propositions are logically independent of one another is spelled out in terms of truth-value assignments: any way of assigning truth values to the elementary propositions is coherent, in the sense that it extends to an assignment of truth values to all propositions that is well-behaved with respect to the truth-functional connectives; e.g., it maps a conjunction to *true* if and only if it maps both conjuncts to *true*. The idea that every proposition can be made from the elementary propositions using the truth-functional operations can be stated in similar terms. For if any given proposition is a truth-functional combination of elementary propositions, then its truth value is completely determined by the truth-values of the elementary propositions. That is to say, there cannot be two different ways of coherently extending an assignment of truth values to the elementary propositions to all of the propositions.

The concepts appealed to here are reminiscent of the concept of a truth-value assignment and valuation found in propositional logic, so we shall repurpose those terms. First, note that up to isomorphism, there is a unique two-element Boolean algebra, which we will call $\mathbf{2}$. As a model of propositions, we can think of this as identifying propositions with truth values: the top element $\top$ is the truth value *true*, and the bottom element $\bot$ is the truth value *false*. The behaviour of $-$ and $\wedge$ conforms to the classical truth tables for negation and conjunction. For mathematical convenience, we identify $\bot$ with 0 and $\top$ with 1; by the von Neumann definition of ordinals, it follows that $2 = \{0, 1\} = \{\bot, \top\}$. A *truth-value assignment* is then a mapping $f : E \to 2$. A *valuation* is a mapping $v : A \to 2$ which respects the truth-functional operations, i.e., such that for all $x, y, z \in A$:

(i) $v(-_{\mathfrak{A}}x) = -_{\mathbf{2}}v(x)$, and

(ii) $v(y \wedge_{\mathfrak{A}} z) = v(y) \wedge_{\mathbf{2}} v(z)$.

We say a valuation v *extends* f when $v(x) = f(x)$ for all $x \in E$.

Definition 2.4. $E \subseteq A$ freely generates $\mathfrak{A}$ with respect to valuations *if every truth-value assignment* $f : E \to 2$ *extends to a unique valuation* $\hat{f} : A \to 2$.

Observe that being freely generated with respect to valuations imposes a non-trivial condition on the granularity of propositions. Propositions must be fine-grained enough to support a well-defined notion of evaluating the truth-value of a proposition in terms of the truth-values of its elementary components. This seems to require propositions at least be structured enough to have a well-defined notion of elementary component. (We can recover the notion of x being definable from a set of elementary propositions X by saying that any two valuations agreeing on every element of X agree on x. When there is a minimal such X we can talk of X being the set of elementary components of x.) Not every Boolean algebra is like this – for instance, we will shortly see that no eight element Boolean algebra is freely generated with respect to valuations by any set of its elements.

The Tractarian picture of propositions as built up from elementary propositions suggests that a wider class of structure-sensitive operations should be well-defined. Consider, now, the operation of taking a proposition – thought of as a truth-functional combination of elementary propositions – and uniformly replacing each elementary constituent with other propositions. This is the "metaphysical" analogue of the operation of substituting the letters of a propositional formula with other formulas, so we might similarly repurpose the word "substitution". If this notion is well-defined, then any function $f : E \to A$ mapping elementary propositions to propositions should extend uniquely to a substitution

$\hat{f} : A \to A$ that informally may be thought of as taking a proposition $x \in A$ and replacing the elementary components according to f, leaving the truth-functional operations in place. It is clear, given this informal idea of leaving the truth-functional operations "in place", that substitutions should commute with the truth-functional connectives, i.e., such that for all $x, y, z \in A$:

(i) $f(-x) = -f(x)$, and

(ii) $f(y \wedge z) = f(y) \wedge f(z)$.

We take these equations as our definition of a substitution. The extendability of functions from E to A to unique substitutions seems, intuitively, to subsume the previous condition: truth-value assignments can be identified with functions $f : A \to A$ that only take on two values, $\top$ and $\bot$.

Definition 2.5. $E \subseteq A$ freely generates $\mathfrak{A}$ with respect to substitutions *if every function $f : E \to A$ extends to a unique substitution $\hat{f} : A \to A$.*

Clearly there is a generalization here to be made. For this, we need another algebraic concept. A *homomorphism* $f : A \to B$ between two Boolean algebras $\mathfrak{A} = \langle A, -_{\mathfrak{A}}, \wedge_{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle B, -_{\mathfrak{B}}, \wedge_{\mathfrak{B}} \rangle$ is a function $f : A \to B$ such that for all $x, y, z \in A$:

(i) $f(-_{\mathfrak{A}} x) = -_{\mathfrak{B}} f(x)$, and

(ii) $f(y \wedge_{\mathfrak{A}} z) = f(y) \wedge_{\mathfrak{B}} f(z)$.

We can now introduce a more general notion of free generation for a given Boolean algebra $\mathfrak{A}$. The more general notion is parametric in two dimensions. The first parameter T is a class of algebras. This parameter delineates the algebras which may serve as targets for the functions on elementary propositions which must be uniquely extendable. The second parameter C consists of a class of Boolean algebras (containing $\mathfrak{A}$ and the algebras in C) along with a class of homomorphisms among these algebras. This parameter imposes constraints on the homomorphism $\hat{f}$ to which a given function f on the elementary propositions must be uniquely extendable. (It would be natural to let C be a category, in the sense of Mac Lane (1998). For our purposes, the additional constraints imposed by categories won't be important, so there is no need to go into the details of such a more rigorous formulation.) Relative to these two parameters, we define:

Definition 2.6. $E \subseteq A$ $\langle \mathsf{T}, \mathsf{C} \rangle$-freely generates $\mathfrak{A}$ *if any function f from E to the elements of a Boolean algebra $\mathfrak{B}$ in T has a unique extension $\hat{f}$ which is a homomorphism in C from $\mathfrak{A}$ to $\mathfrak{B}$.*

In this more general notion, free generation with respect to valuations is $\langle \mathbf{2}, \mathsf{BA} \rangle$-free generation; here, we indicate the singleton class $\{\mathbf{2}\}$ using its single element $\mathbf{2}$ for brevity, and we let BA be the class of Boolean algebras with their homomorphisms. Similarly, free generation with respect to substitutions is $\langle \mathfrak{A}, \mathsf{BA} \rangle$-free generation. The latter requires that any function $f : E \to A$ has a unique extension $\hat{f}$ which is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}$; this is also known as an *endomorphism* on $\mathfrak{A}$. (An endomorphism on $\mathfrak{A}$ which is also a bijection, i.e., an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}$, is know as an *automorphism* of $\mathfrak{A}$.)

We will also consider two further instances of $\langle \mathsf{T}, \mathsf{C} \rangle$-free generation. In our terminology, the strongest notion of free generation in this context – the notion usually used in the theory of Boolean algebras – is the notion of $\langle \mathsf{BA}, \mathsf{BA} \rangle$-free generation. Unpacking the definition, $E \subseteq A$ $\langle \mathsf{BA}, \mathsf{BA} \rangle$-freely generates $\mathfrak{A}$ when any function $f : E \to B$, where B is the underlying set of another Boolean algebra $\mathfrak{B}$, extends to a unique homorophism $\hat{f} : \mathfrak{A} \to \mathfrak{B}$. Intuitively, this corresponds to the idea that the operation of "replacing" elementary propositions with elements of an arbitrary Boolean algebra is well-defined. The second notion is a more modest attempt to strengthen free generation with respect to substitutions. This is $\langle \mathsf{sub}(\mathfrak{A}), \mathsf{BA}\rangle$-free generation, where $\mathsf{sub}(\mathfrak{A})$ is the class of subalgebras of $\mathfrak{A}$. This condition takes us beyond free generation with respect to substitutions since it requires that if you have a function $f : E \to A$, the range of its extension $\hat{f}$ must be included in the subalgebra of $\mathfrak{A}$ generated by the range of f.

2.3 Equivalence

We now have five ways of capturing LA: independent generation and $\langle \mathsf{T}, \mathsf{BA}\rangle$-free generation, for T being $\mathbf{2}$, $\mathfrak{A}$, $\mathsf{sub}(\mathfrak{A})$, or BA. How do they relate? It turns out that they are all equivalent, at least on non-trivial algebras. Indeed, we can show a more general result, which is that independent generation is equivalent, among non-trivial algebras, to $\langle \mathsf{T}, \mathsf{BA}\rangle$-free generation, for every class of Boolean algebras T which contains some non-trivial Boolean algebra. The proof makes use of an algebraic concept which has not been introduced so far. An *ultrafilter* of a Boolean algebra $\langle A, -, \wedge\rangle$ is set $U \subseteq A$ such that $-x \in U$ iff $x \notin U$, and $y \wedge z \in U$ iff $y \in U$ and $z \in U$, for all $x, y, z \in A$. This means that a set $U \subseteq A$ is an ultrafilter just in case its characteristic function, mapping every element of U to $\top$ and every other element of A to $\bot$, is a homomorphism from $\mathfrak{A}$ to $\mathbf{2}$.

Proposition 2.7. *Let $\mathfrak{A} = \langle A, -, \wedge\rangle$ be a non-trivial Boolean algebra, $E \subseteq A$, and T a class of Boolean algebras containing a non-trivial Boolean algebra. Then E independently generates $\mathfrak{A}$ if and only if E $\langle \mathsf{T}, \mathsf{BA}\rangle$-freely generates $\mathfrak{A}$.*

Proof. Assume first that E independently generates $\mathfrak{A}$. As noted in Koppelberg (1989, Proposition 9.4), it follows that E $\langle \mathsf{BA}, \mathsf{BA}\rangle$-freely generates $\mathfrak{A}$. That E $\langle \mathsf{T}, \mathsf{BA}\rangle$-freely generates $\mathfrak{A}$ follows immediately from this.

So, assume now that E $\langle \mathsf{T}, \mathsf{BA}\rangle$-freely generates $\mathfrak{A}$. By assumption, T contains a non-trivial Boolean $\mathfrak{B}$. To show that E is independent, consider any $D \lhd_\omega E$. Define a function $f : E \to B$ such that $f(d) = \top$ if $d \in D$, and $f(d) = \bot$ otherwise. Since $\hat{f}$ is a homomorphism, $\hat{f}(\bigwedge D) = \top$, whence $\bigwedge D > \bot$ as $\mathfrak{A}$ and $\mathfrak{B}$ are non-trivial.

To show that E generates $\mathfrak{A}$, assume otherwise for contradiction. Then $\mathfrak{A}$ has a proper subalgebra $\mathfrak{A}_0$ including E. Consequently, as noted by Givant and Halmos (2009, ch. 20, ex. 15), there exists an ultrafilter U_0 of $\mathfrak{A}_0$ which is extended by distinct ultrafilters U_1, U_2 of $\mathfrak{A}$. Define $f_0 : E \to \{\bot, \top\}$ such that $f_0(e) = \top$ iff $e \in U_0$. Let $f_1, f_2 : A_0 \to \{\bot, \top\}$ such that $f_i(b) = \top$ iff $b \in U_i$, for $i \in \{1, 2\}$. Then f_1 and f_2 are distinct homomorphisms extending f_0, in contradiction with E $\langle \mathsf{T}, \mathsf{BA}\rangle$-freely generating $\mathfrak{A}$. $\square$

What about the trivial Boolean algebra $\mathbf{1}$, based on the one-element set $1 = \{0\}$? It is easy to see that no subset of 1 independently, $\langle \mathsf{BA}, \mathsf{BA}\rangle$-freely, or $\langle \mathbf{2}, \mathsf{BA}\rangle$-freely generates $\mathbf{1}$. In contrast, both $\emptyset$ and 1 $\langle \mathbf{1}, \mathsf{BA}\rangle$-freely (and hence $\langle \mathsf{sub}(\mathbf{1}), \mathsf{BA}\rangle$-freely) generate $\mathbf{1}$. However, for present philosophical purposes, this divergence is relatively uninteresting, simply because the trivial Boolean algebra $\mathbf{1}$ is not a credible model of the algebra of propositions: there is surely at least one truth and at least one falsehood, and so more than one proposition.

Returning to non-trivial algebras, what kinds of pairs E and $\mathfrak{A}$ satisfy independent generation (and so, equivalently, the various notions of free generation)? First, if E independently generates $\mathfrak{A}$, then it does so uniquely, up to isomorphism, in the following sense: if E and E' independently generate $\mathfrak{A}$ and $\mathfrak{A}'$, respectively, and E and E' have the same cardinality, then any bijection between E and E' has a unique extension to an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}'$. Second, any set E independently generates some Boolean algebra $\mathfrak{A}$. Up to isomorphism, we may therefore speak of *the* Boolean algebra which is independently generated by a set of a given cardinality κ. (For a more detailed discussion of these and the following observations, see Givant and Halmos (2009, ch. 28).)

To describe independently generated Boolean algebras in more detail, it is useful to distinguish two cases, depending on whether E is finite or infinite. We begin with the finite case. The *powerset* $\mathcal{P}(W)$ is the set of subsets of a given set W. This forms a Boolean algebra when we add the operations of relative complement and intersection (mapping any $X \subseteq W$ to $W \backslash X$ and any $Y, Z \subseteq W$ to $Y \cap Z$, respectively). We write $\mathfrak{P}(W)$ for this algebra. W itself can be taken to be a powerset $\mathcal{P}(S)$, which gives us the *double powerset algebra* $\mathfrak{P}\mathcal{P}(S)$, whose members are sets of subsets of S. For any $s \in S$, let $s^* = \{w \subseteq S : s \in w\}$. It can be shown that $\Gamma_S = \{s^* : s \in S\}$ independently generates $\mathfrak{P}\mathcal{P}(S)$. (We will see an intuitive explanation of this observation in section 3.3.) We call Γ_S the set of *canonical generators of* $\mathfrak{P}\mathcal{P}(S)$. Further, the function mapping any s to s^* is a bijection from S to Γ_S, whence S and Γ_S have the same cardinality. Thus, any finite set E independently generates a Boolean algebra which is isomorphic to $\mathfrak{P}\mathcal{P}(E)$. So, a finite Boolean algebra is independently generated just in case it is isomorphic to a double powerset algebra.

Consider now the infinite case. One way of generating infinite independently generated Boolean algebras uses a technique commonly employed in logic. Take the language of classical propositional logic, based on a set of proposition letters of some infinite cardinality κ. Quotient this set by provable equivalence: let A be the set of equivalence classes $[\varphi]$ of formulas φ under the relation of provable equivalence in classical propositional logic. This relation of provable equivalence is not only an equivalence relation, but a congruence with respect to the connectives: if φ, ψ, χ are provably equivalent to φ', ψ', χ', respectively, then $\neg\varphi$ and $\psi \wedge \chi$ are also provably equivalent to $\neg\varphi'$ and $\psi' \wedge \chi'$, respectively. Consequently, there is a function $-$ which maps $[\varphi]$ to $[\neg\varphi]$, and a function $\wedge$ which maps $[\psi]$ and $[\chi]$ to $[\psi \wedge \chi]$, for all formulas φ, ψ, χ. $\mathfrak{A}_\kappa = \langle A, -, \wedge \rangle$ is a Boolean algebra, known as the *Lindenbaum-Tarski algebra* (of cardinality κ). $\mathfrak{A}_\kappa$ can be shown to be independently generated by the set E of equivalence classes of proposition letters. No two proposition letters are provably equivalent, so $|E| = \kappa$. Thus, an infinite Boolean algebras is independently generated just in case it is isomorphic to a Lindenbaum-Tarski algebra.

The finite and infinite cases of independently generated Boolean algebras differ sharply in their structural properties. We will shortly see some concrete examples of this, in the form of the properties of atomicity and completeness, introduced below, which finite independently generated Boolean algebras possess but infinite independently generated Boolean algebras lack. A more straightforward observation concerns the relative cardinalities of E and $\mathfrak{A}$: assuming E independently generates $\mathfrak{A}$, if E is finite, then $|A| = 2^{2^{|E|}} > |E|$, whereas if E is infinite, then $|A| = |E|$.

Finally, it is worth illustrating that a set which independently generates a Boolean algebra does not do so uniquely (even though it does so uniquely up to isomorphism). Consider first the finite case, of a double powerset algebra $\mathfrak{P}\mathcal{P}(S)$ based on a finite set S. Any permutation f of $\mathcal{P}(S)$ induces an automorphism $\bar{f}$ of $\mathfrak{P}\mathcal{P}(S)$, where $\bar{f}(x) = \{f(w) : w \in x\}$. Thus, for any such permutation f, the set $\{\bar{f}(x) : x \in \Gamma_S\}$ independently generates $\mathfrak{P}\mathcal{P}(S)$. To make this more concrete, let S be the two-element set $2 = \{0, 1\}$. Then the canonical generators form the following set $\Gamma_2 = \{0^*, 1^*\}$:

$$\{\{\{0\}, \{0, 1\}\}, \{\{1\}, \{0, 1\}\}\}$$

As our permutation f of $\mathcal{P}(2) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$, let us choose the transposition of $\emptyset$ and $\{0\}$, which maps these two elements to each other, and the other two elements to themselves. Then $\bar{f}$ maps the elements of Γ_2 to the elements of the following set, which therefore also independently generates $\mathfrak{P}\mathcal{P}(S)$:

$$\{\{\emptyset, \{0, 1\}\}, \{\{1\}, \{0, 1\}\}\}$$

Similarly, in the case of a Lindenbaum-Tarski algebra, the equivalence classes of proposition letters are not unique as independent generators. A simple example of a distinct set of independent generators is the set of equivalence classes of negations of proposition letters.

3 Complete Boolean Algebras

The equivalence of the five conditions discussed so far suggests that they do well in capturing LA. This coheres with some accounts in the literature, for example Bell and Demopoulos (1996), who operate with the condition of independent generation. However, in the following, we will consider a more restricted class of Boolean algebras, and strengthened versions of the five conditions. We begin with one possible way of motivating these restrictions.

3.1 Quantification

Logical atomists have to find a way of accommodating quantification. If there are finitely many individuals, this is easily done using conjunction and disjunction. For example, the proposition that everything is F can be identified with the conjunction of propositions Fx, for every individual x (cf. LPA, 5.52). Existential quantification can be treated analogously, using disjunction. However, if there are infinitely many individuals, this idea will not obviously work, since Boolean algebras only provide finitary truth-functional operations.

One way of accommodating quantification is therefore to provide infinitary analogs of conjunction and disjunction. This can be done by assuming the algebra of propositions to be complete. A Boolean algebra is *complete* if every set X of elements has a greatest lower bound $\bigwedge X$ and a least upper bound $\bigvee X$. Just as the greatest lower bound of two elements $x \wedge y$ can be understood as their conjunction, the greatest lower bound of a set of elements $\bigwedge X$ can be understood as their (possibly infinite) conjunction. The same applies to disjunction and least upper bounds.

The need to accommodate quantifiers therefore motivates capturing LA using only complete Boolean algebras. Further, the availability of potentially infinitary truth-functional operations suggests adapting the five conditions discussed above, taking into account not just the binary operation $\wedge$, but the more general operation $\bigwedge$ on sets. ($\bigvee$ can be defined in terms of $-$ and $\bigwedge$, just as $\vee$ can be defined in terms of $-$ and $\wedge$.)

3.2 Regimentations, Revised

Let us therefore revisit the regimentations of LA discussed above in the setting of complete Boolean algebras. First, we adjust the definition of finite descriptions to take into account the possibility of conjoining arbitrary sets of propositions. This allows us to consider complete descriptions, which decide, for each elementary proposition, whether to include it or its negation. This gives rise to a notion of a set being completely independent:

Definition 3.1. *D is a* complete description based on *E, written $D \lhd E$, if $D = T \cup \{-f : f \in F\}$ for disjoint sets $T, F \subseteq E$ such that $T \cup F = E$. E is* completely independent *if $\bigwedge D > \bot$ for every $D \lhd E$.*

We adapt the notion of a subalgebra as follows: $\langle B, -', \wedge' \rangle$ is a *complete subalgebra* of a complete Boolean algebra $\langle A, -, \wedge \rangle$ if it is a subalgebra of $\mathfrak{A}$ which preserves greatest lower bounds (and therefore least upper bounds), in the following sense: for any $X \subseteq B$, the greatest lower bound of X in $\mathfrak{B}$ is the greatest lower bound in $\mathfrak{A}$. This gives us a complete notion of generation:

Definition 3.2. $E \subseteq A$ completely generates $\mathfrak{A}$ *if the smallest complete subalgebra of $\mathfrak{A}$ including E is $\mathfrak{A}$ itself.*

With this, we can adapt the notion of independent generation by defining, for any complete Boolean algebra $\mathfrak{A}$:

Definition 3.3. $E \subseteq A$ completely independently generates $\mathfrak{A}$ *if* E *is completely independent, and completely generates* $\mathfrak{A}$*.*

Moving on to the other notions of free generation, let a *complete homomorphism* from $\langle A, -, \wedge\rangle$ to $\langle B, -', \wedge'\rangle$ (both complete Boolean algebras) be a homomorphism f which preserves greatest lower bounds (and therefore least upper bounds), in the following sense: for any $X \subseteq A$, f maps the greatest lower bound of X in $\mathfrak{A}$ to the greatest lower bound of $\{f(x) : x \in X\}$ in $\mathfrak{B}$. (Correspondingly, a *complete endomorphism* on an algebra $\mathfrak{A}$ is a complete homomorphism from $\mathfrak{A}$ to $\mathfrak{A}$.) With this, let CBA be the class of complete Boolean algebras along with the class of complete homomorphisms among complete Boolean algebras.

We can now instantiate the notion of $\langle \mathsf{T}, \mathsf{C}\rangle$-free generation by letting C be CBA, and T be one of the four relevant classes of complete Boolean algebras. So, we will be interested in $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free, $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-free, $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free, and $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-free generation, where $\mathsf{csub}(\mathfrak{A})$ is the class of complete subalgebras of $\mathfrak{A}$.

We will aim to characterize these conditions in more direct, structural terms, and to delineate how they relate to each other. We have been able to provide structural characterizations of three of the five conditions: complete independent generation, $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation, and $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-free generation. These will be presented in section 4. The remaining two conditions of $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-free generation and $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation are more difficult to understand, and we have only been able to describe partially how the relate to each other and the other three notions. The relevant results will be discussed in section 5. Before delving into these mathematical questions, it is worth mentioning a closely related way of regimenting logical atomist ideas, which also uses certain complete Boolean algebras.

3.3 Possible Worlds

In the literature on logical atomism, some authors have also adopted (complete) Boolean algebras, but imposed a condition which is at least apparently different from those considered above. In particular, Suszko (1968), Moss (2012), and Button (2017) effectively suggest modeling the logical atomist account of propositions using double powerset algebras, as defined above. Somewhat less explicitly, one may also find this understanding of logical atomism in Ramsey (1923, 1927).

Why double powerset algebras? First, powerset algebras provide very natural models of propositions, using the ideas of possible world semantics. Think of a set W as the set of possible worlds, and assume that the propositions are the sets of possible worlds, identifying any proposition with the set of possible worlds in which it is true (cf. LPA, 4.2, 4.4, 4.431). Since a proposition is true in a world just in case its negation is not true there, relative complement serves as the operation of negation. Similarly, a conjunctive proposition is true in a world just in case its conjuncts are true in this world; consequently, intersection serves as the operation of conjunction. Thus, on this picture, we can identify the propositional algebra with the powerset algebra $\mathfrak{P}(W)$.

Second, from this perspective of possible world semantics, it is a natural to capture LA by letting the elementary propositions determine the possible worlds. That is, taking elementary propositions again to be logically independent, it is natural to think that for any set of elementary propositions X, there is a possible world in which every member of X is true, and every elementary proposition not in X is false. Further, since all propositions are truth-functional combinations of elementary propositions, the truth of every proposition should be determined by the truth and falsity of elementary propositions. Thus, there should only be one world in which every member of X is true and every elementary proposition not in X is false. Consequently, we should be able to identify every possible world with the set of elementary propositions true in it; more generally, we should be able to identify the possible worlds W with the sets of elementary propositions (cf. LPA, 4.27, 4.28, 4.45).

Putting these ideas together, we arrive at double powerset algebras. To be precise, we have to be a bit careful about the status of elementary propositions. By way of example, consider an elementary proposition $e \in E$. The informal ideas sketched above suggested that the set of worlds in which e is true is $e^* = \{w \subseteq E : e \in w\}$. Since e is a proposition, it has to be the set of worlds in which it is true, and so $e = e^*$. But $e \in \{e\} \in e^*$, whence $e \neq e^*$. Therefore, we should strictly speaking start from a set of, say, elementary *states* S, with the set of worlds $W = \mathcal{P}(S)$. Each state s then corresponding uniquely to an elementary *proposition* $s^* = \{w \subseteq S : s \in w\}$, the proposition that state s obtains. We thus arrive at the elementary propositions forming the set $E = \{s^* : s \in S\}$ of elements of the double powerset algebra of propositions $\mathfrak{P}\mathcal{P}(S)$.

Among finite algebras, we have seen above that this regimentation of logical atomism (relying on the possible worlds account of propositions) fits exactly the condition of independent generation and its equivalent formulations in terms of different kinds of free generation, at least up to isomorphism. However, among infinite algebras, it diverges sharply from these conditions, since no infinite powerset algebra is independently generated. This follows from the fact that infinite independently generated Boolean algebras are incomplete, while all powerset algebras are complete, with intersections and disjunctions serving as greatest lower bounds and least upper bounds.

The divergence can also be shown by considering the atoms of algebras. An *atom* of a Boolean algebra $\mathfrak{A}$ is a non-zero element a for which there is no x such that $\bot < x < a$. $\mathfrak{A}$ is *atomless* if it has no atoms, and *atomic* if for every non-zero element x, there is some atom $a \leq x$. Every powerset algebra is atomic, with the singleton sets serving as atoms. Indeed, a Boolean algebra is isomorphic to a powerset algebra if and only if it is both complete and atomic. In contrast, every infinite free Boolean algebra is not only incomplete but also atomless.

Intuitively, it is not that surprising that the double powerset condition comes apart so decisively from the conditions formulated above. The reason is that in the double powerset condition, possible worlds are essentially generated from elementary propositions by taking conjunctions of elementary propositions and their negations. If there are infinitely many elementary propositions, then these conjunctions have to be infinite. As noted, Boolean algebras which are not complete will not in general provide such an infinitary operation of conjunction. Even if the algebra under consideration is complete, the conditions formulated above do not take into account any infinitary truth-functional operations.

These observations suggest an intriguing possibility: Could it be that independent generation and its variant definitions in terms of free generation come to match the double powerset condition when they are adjusted to accommodate infinitary operations? We will see below that this is not the case, although some of the conditions come close. In general, we will see that the mathematical situation becomes much more complicated once we move to complete Boolean algebras in the way sketched here. Starting to explore this territory will be the main task for the rest of this paper. In the following, results concerning E and $\mathfrak{A}$ assume implicitly that E is a set of elements of a non-trivial complete Boolean algebra $\mathfrak{A}$.

4 Structural Characterizations

The following three sections provide structural characterizations of $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-free generation, $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation, and complete independent generation, respectively.

4.1 $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-Free Generation

We begin with $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-free generation, since it can be characterized using a very simple condition. The complete Boolean algebras which are $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-freely generated are just the finite double powerset algebras. To state this more carefully, we write again e^* for $\{w \subseteq E : e \in w\}$, where e is an element of a contextually salient set E, and we continue to use this notation in the following.

Theorem 4.1. *E $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$ just in case E is finite and there is an isomorphism f from $\mathfrak{A}$ to $\mathfrak{P}\mathcal{P}(E)$ such that $f(e) = e^*$ for all $e \in E$.*

Proof. Among finite algebras, $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-free generation coincides with $\langle \mathsf{BA}, \mathsf{BA}\rangle$-free generation, i.e., independent generation. So, in these cases, the claim follows from the characterization of independent generation in section 2.3. It remains to show that if $\mathfrak{A}$ is infinite, it is not $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-freely generated by any set E. This follows from a result of Gaifman (1964) and Hales (1964), who show that for every cardinality κ, there is a complete Boolean algebra of size $> \kappa$ which is completely generated by a countable set. (An elegant proof was provided by Solovay (1966); see also Koppelberg (1989, p. 191, Corollary 13.2).) The required corollary of the Gaifman-Hales result is straightforward and well-known – see the title of Hales (1964) – but since it is central for this paper, we spell out the argument:

Assume for contradiction that $\mathfrak{A}$ is infinite, and $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-freely generated by E. Along the lines of the proof of Proposition 2.7, is easy to see that E must be infinite, since the subalgebra (completely) generated by any finite set is finite. By the result of Gaifman and Hales, there is a complete Boolean algebra $\mathfrak{B}$ which is completely generated by a countable set G, while being of greater cardinality than $\mathfrak{A}$. Since E is infinite, there is a surjective function $f : E \to G$. By $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-free generation, f can be extended to a complete homomorphism $\hat{f}$ from $\mathfrak{A}$ to $\mathfrak{B}$. Since $\hat{f}$ is surjective on G, and G completely generates $\mathfrak{B}$, it follows by a transfinite induction that $\hat{f}$ is also surjective on $\mathfrak{B}$. But this is impossible, since $\mathfrak{B}$ is of larger cardinality than $\mathfrak{A}$. □

$\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-free generation is thus inconsistent with the existence of an infinity of propositions, and so is implausible insofar as it is plausible that there are infinitely many propositions (cf. LPA, 4.2211). The Gaifman-Hales theorem also sheds some light on the notion of complete generation: If $\mathfrak{A}$ is generated by E, then every element of A can be obtained from E by a *finite* sequence of applications of $-$ and $\wedge$ to the members of E. If $\mathfrak{A}$ is *completely* generated by E, then to construct an arbitrary member of A, we may have to use applications of $\bigwedge$ to infinite sets. However, one might naturally wonder whether we may also need to iterate the applications of $-$ and $\bigwedge$ an infinite number of times. That this is the case follows from Gaifman-Hales. To make this precise, we define, analogous to section 2.1:

$$
\begin{aligned}
E_0^\infty &:= E \\
E_{n+1}^\infty &:= \{-x : x \in E_n^\infty\} \cup \{\textstyle\bigwedge Y : Y \subseteq E_n^\infty\} \\
E_\omega^\infty &:= \bigcup_{n\in\mathbb{N}} E_n^\infty
\end{aligned}
$$

By construction, the cardinality of E_ω^∞ cannot be greater than $\beth_\omega = \sup\{\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots\}$. By Gaifman-Hales, $\mathfrak{A}$ may be generated by a countable set G, but be of larger cardinality than $\beth_\omega$. The construction of generated elements may therefore have to be iterated into the transfinite (see the proof of Lemma 4.5). Of course, this is a consequence of the fact that only $-$ and $\bigwedge$ are treated as primitive here. If in constructing elements of complete Boolean algebras we allowed also $\bigvee$ and all the other possible operations in complete Boolean algebras, every element of A could trivially be constructed from G in just one step (cf. LPA, 5.3, 5.32).

4.2 $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free Generation

For much of the following, we need two more basic notions for Boolean algebras, namely relativization and product. First, let $\langle A, -, \wedge\rangle$ be a Boolean algebra, and $r \in A$. The *relativization of* $\mathfrak{A}$ *to* r is the Boolean algebra $\langle B, -', \wedge'\rangle$ where $B = \{y \in A : y \leq r\}$, $-'x = r \wedge -x$, and $y \wedge' z = y \wedge z$, for all $x, y, z \in B$. Second, if $\langle A, -_{\mathfrak{A}}, \wedge_{\mathfrak{A}}\rangle$ and $\langle B, -_{\mathfrak{B}}, \wedge_{\mathfrak{B}}\rangle$ are Boolean algebras, then $\mathfrak{A} \times \mathfrak{B}$, the *product of* $\mathfrak{A}$ *and* $\mathfrak{B}$, is the Boolean algebra $\langle A \times B, -_{\mathfrak{A}\times\mathfrak{B}}, \wedge_{\mathfrak{A}\times\mathfrak{B}}\rangle$ such that, for all $x, y, z \in A$ and $x', y', z' \in B$:

$$-_{\mathfrak{A}\times\mathfrak{B}}\langle x, x'\rangle = \langle -_{\mathfrak{A}}x, -_{\mathfrak{B}}x'\rangle$$

$$\langle y, y'\rangle \wedge_{\mathfrak{A}\times\mathfrak{B}} \langle z, z'\rangle = \langle y \wedge_{\mathfrak{A}} y', z \wedge_{\mathfrak{B}} z'\rangle$$

For brevity, we will often use $-$ and $\wedge$ for the two operations of any Boolean algebra, letting context disambiguate to which algebra the relevant operation belongs.

In the following, we will rely on a basic observation about products of complete Boolean algebras, namely that any complete Boolean algebra $\mathfrak{A}$ is isomorphic to a product $\mathfrak{B} \times \mathfrak{C}$ of a complete atomless Boolean algebra $\mathfrak{B}$ and a complete atomic Boolean algebra $\mathfrak{C}$. More specifically, let x be the least upper bound of the set of atoms in $\mathfrak{A}$. Let $\mathfrak{B}$ be the relativization of $\mathfrak{A}$ to $-x$, and $\mathfrak{C}$ the relativization of $\mathfrak{A}$ to x. Then $\mathfrak{B}$ is complete and atomless, $\mathfrak{C}$ is complete and atomic, and $\mathfrak{A}$ is isomorphic to $\mathfrak{B} \times \mathfrak{C}$. (See Givant and Halmos (2009, p. 227, Corollary 2). To see why this observation applies in particular to atomic and atomless Boolean algebras, note that the trivial (one-element) Boolean algebra $\mathbf{1}$ is both atomic and atomless.)

We first note that double powerset algebras are $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generated by their canonical generators, in the finite as well as in the infinite case. Indeed, we can expand this observation to any product of a complete atomless Boolean algebra with a double powerset algebra, turning each canonical generator of the double powerset algebra into a generator of the product algebra by adjoining an arbitrary element of the atomless algebra. The proof of this observation appeals to another basic notion and observation which will remain important in the following: An ultrafilter U is *principal* if it has a least element, i.e., if $\bigwedge U \in U$. Equivalently, a set $U \subseteq A$ is a principal ultrafilter just in case its characteristic function (mapping every element of U to $\top$ and every other element of A to $\bot$) is a complete homomorphism from $\mathfrak{A}$ to $\mathbf{2}$. Furthermore, the principal ultrafilters of a Boolean algebra correspond uniquely to the atoms of the algebra; in particular, if U is a principal ultrafilter, then $\bigwedge U$ is the corresponding atom, and if a is an atom, then $\{x : x \geq a\}$ is the corresponding principal ultrafilter.

Lemma 4.2. *For any complete atomless Boolean algebra* $\mathfrak{B}$*, set* S*, and function* $\beta : S \to B$*,* $\mathfrak{A} = \mathfrak{B} \times \mathfrak{P}\mathcal{P}(S)$ *is* $\langle \mathbf{2}, \mathsf{CBA}\rangle$*-freely generated by* $E = \{\langle \beta(s), s^*\rangle : s \in S\}$*.*

Proof. Consider any function $f : E \to 2$. Let $\Sigma = \{s \in S : f(\langle \beta(s), s^*\rangle) = 1\}$. Define $\hat{f} : A \to 2$ such that:

$$\hat{f}(\langle b, c\rangle) = 1 \text{ iff } \Sigma \in c$$

It is straightforward to verify that $\hat{f}$ is a complete homomorphism extending f.

For uniqueness, consider any complete homomorphism g extending f. Since g is a complete homomorphism, the set $U \subseteq A$ of elements mapped by g to 1 forms a principal ultrafilter. Thus, g maps the atom $a = \bigwedge U$ to 1. By construction of $\mathfrak{A}$, a must be identical to $\langle \bot_{\mathfrak{B}}, \{w\}\rangle$, for some $w \subseteq S$. We show that $w = \Sigma$. Consider any $s \in S$: $f(\langle \beta(s), s^*\rangle) = 1$ iff $\langle \beta(s), s^*\rangle \in U$ iff $\langle \bot_{\mathfrak{B}}, \{w\}\rangle \leq_{\mathfrak{A}} \langle \beta(s), s^*\rangle$ iff $\{w\} \subseteq s^*$ iff $w \in s^*$ iff $s \in w$. Thus $w = \Sigma$. With this, it follows that $\hat{f}(\langle b, c\rangle) = 1$ iff $w \in c$ iff $\langle \bot_{\mathfrak{B}}, \{w\}\rangle \leq \langle b, c\rangle$ iff $\langle b, c\rangle \in U$ iff $g(\langle b, c\rangle) = 1$. So $\hat{f} = g$. □

Next, we note that up to isomorphism, these are the only examples of $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation. To state this result, we assume the convention of writing π_i for the i-th projection function, which maps any n-tuple with $n \geq i$ to its ith coordinate. E.g., $\pi_2(\langle x, y\rangle) = y$.

Lemma 4.3. *If E $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$, then there is a complete atomless Boolean algebra $\mathfrak{B}$ and an isomorphism f from $\mathfrak{A}$ to $\mathfrak{B} \times \mathfrak{P}\mathcal{P}(E)$ such that $\pi_2 f(e) = e^*$ for all $e \in E$.*

Proof. Assume E $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$. Let c be the least upper bound of the set of atoms of $\mathfrak{A}$, $b = -c$, and $\mathfrak{B}$ the relativization of $\mathfrak{A}$ to b. As noted above, $\mathfrak{B}$ is complete and atomless. For every $w \subseteq E$, let $D_w = w \cup \{-e : e \in E\backslash w\}$. Define a function f on A such that for all $x \in A$:

$$f(x) = \langle x \wedge b, \{w \subseteq E : \bigwedge D_w \wedge c \leq x\}\rangle$$

We show that $\mathfrak{B}$ and f witness the claim.

First, we show that for every $w \subseteq E$, $\bigwedge D_w \wedge c$ is an atom of $\mathfrak{A}$. Let g be the characteristic function of w, mapping any $e \in w$ to 1 and $e \in E\backslash w$ to 0. By $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation, g has a unique extension $\hat{g}$ to a complete homomorphism from $\mathfrak{A}$ to $\mathbf{2}$. Let U be the corresponding ultrafilter, the set of elements mapped by $\hat{g}$ to 1. Since $\hat{g}$ is complete, U is principal, whence $a = \bigwedge U$ is an atom. Further, $g(x) = 1$ for all $x \in D_w$, and so $\hat{g}(\bigwedge D_w) = 1$. Thus, $a \leq \bigwedge D_w$, and so $a \leq \bigwedge D_w \wedge c$. We show that $a = \bigwedge D_w \wedge c$. Assume otherwise for contradiction. Then there is another atom $a' \leq \bigwedge D_w \wedge c$, and so a corresponding principal ultrafilter $U' = \{x \in A : a' \leq x\}$. Let h be the characteristic function of U', mapping every $x \in U'$ to 1 and $x \in A\backslash U'$ to 0. Then h is a complete homomorphism from $\mathfrak{A}$ to $\mathbf{2}$. Further, since $a' \leq \bigwedge D_w$, $D_w \subseteq U'$, so h extends g. Thus h contradicts the uniqueness of $\hat{g}$.

With this observation, it is straightforward to verify that f is a homomorphism. To establish that f is surjective, it suffices to show that any $\langle x, y\rangle \in B \times \mathcal{PP}(E)$ is the image of $x \vee \bigvee_{w \in y} \bigwedge D_w$ under f, which is routine. To establish that f is injective, consider any distinct members x and y of A. Then either $x \wedge b \neq y \wedge b$ or $x \wedge c \neq y \wedge c$. In the former case, it is immediate that $\pi_1 f(x) \neq \pi_1 f(y)$. In the latter case, we may assume without loss of generality that there is an atom $a \leq x$ such that $a \not\leq y$. Let $w = \{e \in E : a \leq e\}$. As established above, $\bigwedge D_w \wedge c$ is an atom, and so $\bigwedge D_w \wedge c = a$. Thus $\bigwedge D_w \wedge c \not\leq y$. Thus, w witnesses $\pi_2 f(x) \neq \pi_2 f(y)$. So, in either case, $f(x) \neq f(y)$.

Finally, consider any $e \in E$, with a view to establishing $\pi_2 f(e) = e^*$. Let $w \subseteq E$; it suffices to show that $\bigwedge D_w \wedge c \leq w$ iff $w \in e^*$, i.e., iff $e \in w$. If $e \in w$, the claim is immediate. Otherwise, $-e \in D_w$. As shown above, $\bigwedge D_w \wedge c$ is an atom, and so non-zero; thus, $\bigwedge D_w \wedge c \not\leq e$, as required. □

Putting these two lemmas together, we obtain the following characterization of $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation:

Theorem 4.4. *E $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$ just in case there is a complete atomless Boolean algebra $\mathfrak{B}$ and an isomorphism f from $\mathfrak{A}$ to $\mathfrak{B} \times \mathfrak{P}\mathcal{P}(E)$ such that $\pi_2 f(e) = e^*$ for all $e \in E$.*

Proof. The left to right direction is established in Lemma 4.3. The right to left direction follows from Lemma 4.2 and the fact that $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation is invariant under isomorphism. □

4.3 Complete Independent Generation

To characterize complete independent generation, we begin with a basic observation on complete descriptions based on complete generators of complete algebras. The observation is based on the fact that as in the finite case, we can characterize the smallest complete subalgebra of a given algebra containing a starting set of elements as the result of successively closing the starting set under complementation and greatest lower bounds, now admitting arbitrary greatest lower bounds and iterating the construction into the transfinite:

Lemma 4.5. *Let $\mathfrak{A}$ be a complete Boolean algebra which is completely generated by a set $G \subseteq A$, and $D \lhd A$. Then $\bigwedge D \leq x$ or $\bigwedge D \leq -x$ for all $x \in A$.*

Proof. For every ordinal α, define a set $G_\alpha \subseteq A$ as follows:

$$G_0 := G$$
$$G_{\alpha+1} := \{-x : x \in G_\alpha\} \cup \{\bigwedge X : X \subseteq G_\alpha\}$$
$$G_\lambda := \bigcup_{\alpha<\lambda} G_\alpha \quad (\lambda \text{ a limit ordinal})$$

By cardinality considerations, there is some ordinal α at which G_α reaches a fixed point. This set is closed under the two operations, and so a complete subalgebra, which means that $A = G_\alpha$. It thus suffices to show by a transfinite induction that for every ordinal α and $x \in G_\alpha$, $\bigwedge D \leq x$ or $\bigwedge D \leq -x$, which is routine. □

We can now characterize complete independent generation in very similar terms to those just used to characterize $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation. First, we can again show that a product of a complete atomless Boolean algebra $\mathfrak{B}$ with a double powerset algebra $\mathfrak{P}\mathcal{P}(S)$ is completely independently generated, with one additional cardinality constraint: the atomless factor $\mathfrak{B}$ must be generated by a set G which is no larger than the base set S on which the double powerset algebra is constructed. To produce the witnessing set E of completely independent complete generators, we just need to adjoin to every canonical generator s^* of the double powerset algebra an element $\beta(s)$ of G, ensuring that the image of S under β contains each element of G. In other words:

Lemma 4.6. *For any complete atomless Boolean algebra $\mathfrak{B}$, set S, and function $\beta : S \to B$ such that $\{\beta(s) : s \in S\}$ completely generates $\mathfrak{B}$, $\mathfrak{A} = \mathfrak{B} \times \mathfrak{P}\mathcal{P}(S)$ is completely independently generated by $E = \{\langle \beta(s), s^*\rangle : s \in S\}$.*

Proof. We begin by showing that the atoms of $\mathfrak{A}$ can be described as the set $\{\bigwedge D : D \lhd E\}$. Note that $\{D : D \lhd E\} = \{D_w : w \subseteq S\}$, where D_w is defined as $w \cup \{-s : s \in S\backslash w\}$. It thus suffices to show that $\bigwedge D_w = \langle \bot, \{w\}\rangle$, for all $w \subseteq S$. We consider the two coordinates in turn.

First, working in the atomless algebra $\mathfrak{B}$: If $w \subseteq S$, then $\{\pi_1(x) : x \in D_w\}$ includes some $D \lhd G = \{\beta(s) : s \in S\}$. By Lemma 4.5, for all $x \in B$, $\bigwedge D \leq x$ or $\bigwedge D \leq -x$. Since $\mathfrak{B}$ is atomless, it follows that $\bigwedge D = \bot$, and so $\bigwedge\{\pi_1(x) : x \in D_w\} = \bot$, as required.

Second, working in the double powerset algebra $\mathfrak{P}\mathcal{P}(S)$: If $w \subseteq S$, then $D := \{\pi_2(x) : x \in D_w\} \lhd \Gamma_S$. In particular,

$$D = \{s^* : s \in w\} \cup \{-s^* : s \in S\backslash w\}.$$

It is routine to verify that for all $v \subseteq S$, $v \in \bigcap D$ iff $v = w$. Thus $\bigcap D = \{w\}$, and so $\bigwedge\{\pi_2(x) : x \in D_w\} = \{w\}$, as required.

From this initial observation, we can immediately conclude that E is completely independent: for every $D \lhd E$, $\bigwedge D$ is an atom, and so non-zero.

It is also easy to conclude from this observation that E completely generates $\mathfrak{A}$: Let $\mathfrak{A}'$ be the smallest complete subalgebra of $\mathfrak{A}$ including E. By the initial observation, $\mathfrak{A}'$ contains every atom of $\mathfrak{A}$. Since every member of a powerset algebra is a union (least upper bound) of some of its atoms, it follows that $\langle \bot, y\rangle \in \mathfrak{A}'$, for all $y \in \mathcal{PP}(S)$. In particular, $\mathfrak{A}'$ contains $\langle \bot, \top\rangle$, and so also $\langle \beta(s), \bot\rangle$, for all $s \in S$. Since G completely generates $\mathfrak{B}$, it follows that $\mathfrak{A}'$ also contains $\langle x, \bot\rangle$, for all $x \in B$. Thus, $\mathfrak{A}'$ contains any $\langle x, y\rangle \in \mathfrak{A}$, as required. □

As in the case of $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation, we can show that up to isomorphism, these are the only examples of complete independent generation:

Lemma 4.7. *If E completely independently generates $\mathfrak{A}$, then there is a complete atomless Boolean algebra $\mathfrak{B}$ and an isomorphism f from $\mathfrak{A}$ to $\mathfrak{B} \times \mathfrak{P}\mathcal{P}(E)$ such that $\{\pi_1 f(e) : e \in E\}$ completely generates $\mathfrak{B}$ and $\pi_2 f(e) = e^*$ for all $e \in E$.*

Proof. The proof follows the pattern of Lemma 4.3. Assume E completely independently generates $\mathfrak{A}$. Let c be the least upper bound of the set of atoms of $\mathfrak{A}$, $b = -c$, and $\mathfrak{B}$ the relativization of $\mathfrak{A}$ to b. As noted above, $\mathfrak{B}$ is complete and atomless. For every $w \subseteq E$, let $D_w = w \cup \{-e : e \in E \backslash w\}$. Define a function f on A such that for all $x \in A$:

$$f(x) = \langle x \wedge b, \{w \subseteq E : \bigwedge D_w \leq x\}\rangle$$

We show that $\mathfrak{B}$ and f witness the claim.

First, we show that for every $w \subseteq E$, $\bigwedge D_w$ is an atom of $\mathfrak{A}$. That $\bigwedge D_w > \bot$ is guaranteed by the complete independence of E. As E completely generates $\mathfrak{A}$, it therefore follows with Lemma 4.5 that D_w is an atom. The remainder of the proof proceeds along the lines of the proof of Lemma 4.3. □

Putting the last two lemmas together, we obtain the following characterization of complete independent generation:

Theorem 4.8. *E completely independently generates $\mathfrak{A}$ just in case there is a complete atomless Boolean algebra $\mathfrak{B}$ and an isomorphism f from $\mathfrak{A}$ to $\mathfrak{B} \times \mathfrak{P}\mathcal{P}(E)$ such that $\{\pi_1 f(e) : e \in E\}$ completely generates $\mathfrak{B}$ and $\pi_2 f(e) = e^*$ for all $e \in E$.*

Proof. The left to right direction is established in Lemma 4.7. The right to left direction follows from Lemma 4.6 and the fact that complete independent generation is invariant under isomorphism. □

One obvious consequence of Theorems 4.4 and 4.8 is that by moving from arbitrary Boolean algebras to complete Boolean algebras, we include not only the finite double powerset algebras, but also infinite double powerset algebras. It is worth noting that in this case, the relevant generators need not be determined uniquely up to isomorphism. Recall how we noted that for any cardinality κ, there is a Boolean algebra $\mathfrak{A}$ which is independently generated by a set E of cardinality κ. Furthermore, in this case, $\mathfrak{A}$ and E determine each other uniquely, up to isomorphism: $\mathfrak{A}$ is uniquely determined, up to isomorphism, by the cardinality of E, and the cardinality of E is uniquely determined by the structure – indeed, by the cardinality – of $\mathfrak{A}$. In the case of double powerset algebras, it is obvious that the cardinality of the starting set S, which is identical to the cardinality of the set of canonical generators Γ_S, determines $\mathfrak{P}\mathcal{P}(S)$ uniquely, up to isomorphism. If S is finite, its cardinality is also determined by the cardinality of $\mathfrak{P}\mathcal{P}(S)$, since there is a unique number n such that $2^{2^n} = |\mathfrak{P}\mathcal{P}(S)|$. This need not hold if S is infinite, since it is consistent with ZFC that there are distinct cardinals κ, λ such that $2^\kappa = 2^\lambda$. (This consistency claim can be obtained using forcing, the standard technique for independence proofs in set theory. For example, it follows as a corollary of Easton's theorem; see Jech (2002, p. 232, Theorem 15.18).) In this case, $\mathfrak{P}\mathcal{P}(\kappa)$ is isomorphic to $\mathfrak{P}\mathcal{P}(\lambda)$ while $\Gamma_\kappa = \kappa \neq \lambda = \Gamma_\lambda$. In such a case, there is a double powerset algebra which is, e.g., completely independently generated by two sets of distinct cardinalities. Thus, the relevant regimentations of LA admitting all double powerset algebras need not even pin down the elementary propositions up to isomorphism.

Another obvious consequence of Theorems 4.4 and 4.8 is that neither of the relevant constraints requires the generated algebra to be a double powerset algebra. In the case of $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation, it is perhaps not all that surprising that some complete Boolean algebra which is not isomorphic to a double powerset algebra is $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generated by one of its sets. After all, $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation only captures the idea that the *truth-values* of all propositions are uniquely determined by the truth-values of the elementary propositions. The matter is perhaps more surprising in the case of complete independent generation. Along lines discussed in the proof of Lemma 4.7, if E completely independently generates $\mathfrak{A}$, then the atoms of $\mathfrak{A}$ correspond uniquely to the subsets of E, via the greatest lower bounds of complete descriptions in terms of E. This is as one might expect. The (arbitrary) disjunctions of atoms naturally form a double powerset structure. However, we have seen that it is consistent with complete independent generation that further elements are generated: $\mathfrak{A}$ might be the product of an atomless algebra $\mathfrak{B}$ with a double powerset algebra $\mathfrak{P}\mathcal{P}(S)$. In this case, every generating element consists of a component of the atomless algebra and a component of the double powerset algebra. When we form the atoms as greatest lower bounds of complete descriptions in terms of E, the atomless components disappear. Thus, once we have formed the disjunction of all atoms, we can use this to access the atomless component of the generating elements, and so generate the rest of the algebra. Thus, aside from the expected double powerset structure containing $2^{2^{|E|}}$ elements, complete independent generation allows the generation of further, maybe unexpected, propositions, based on the atomless factor $\mathfrak{B}$. It is worth noting that with the Gaifman-Hales theorem, it follows immediately that this atomless factor may be arbitrarily large: If E is countably infinite, the corresponding double powerset algebra will have cardinality $2^{2^{|E|}} = 2^{2^{\aleph_0}} = \beth_2$. But since there are arbitrarily large complete Boolean algebras completely generated by a countable set, the atomless factor may be arbitrarily large. The unexpected atomless factor may therefore dwarf the expected double powerset factor by an arbitrarily large degree.

So, in the case of $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free and completely independent generation at least, the rapprochement with the double powerset picture is incomplete. Naturally, one can close the gap by making further assumptions, such as atomicity: it follows immediately from Theorem 4.8 that a complete *atomic* Boolean algebra $\mathfrak{A}$ is, e.g., completely independently generated by a set E just in case there is an isomorphism from $\mathfrak{A}$ to $\mathfrak{P}\mathcal{P}(E)$ which maps every $e \in E$ to e^*. Thus, to arrive at the double powerset picture, it also suffices to assume, in addition to completely independently generated, that the disjunction of the conjunctions of complete descriptions in terms of E is the top element, i.e.:

$$\bigvee_{D \lhd E} \bigwedge D = \top$$

Such additional assumptions are not obviously part of LA, the principle that every proposition is a truth-functional combination of elementary propositions, although they may be part of a more general logical atomist picture (cf. LPA, 4.46, 4.461).

5 Partial Characterizations

For an arbitrary complete Boolean algebra $\mathfrak{A}$, the notions of $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-free and $\langle\mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation are much harder to understand, and we have only obtained partial results characterizing them and relating them to three notions discussed in the previous section. Our main results are summarized in Figure 1. There, indexed arrows indicate the strongest relationship we have been able to establish. For example, $X \xrightarrow{\subseteq} Y$ indicates that we have been able to establish that X entails Y, but that we haven't been able to settle whether Y entails X. Complete independent generation is abbreviated as CIG, and $\langle\mathsf{T}, \mathsf{CBA}\rangle$-free generation is abbreviated as $\langle\mathsf{T}, \mathsf{CBA}\rangle$, for the various choices of T.

$$
\begin{tikzcd}
 & & \text{CIG} \arrow[r, "\subsetneq"] \arrow[dd, "\not\subseteq"] & \langle \mathbf{2}, \mathsf{CBA}\rangle \\
\langle \mathsf{CBA}, \mathsf{CBA}\rangle \arrow[r, "\subsetneq"] & \langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle \arrow[ur, "\subsetneq"] \arrow[dr, "\subseteq"'] & & \\
 & & \langle \mathfrak{A}, \mathsf{CBA}\rangle &
\end{tikzcd}
$$

Figure 1: Entailments among complete notions of generation.

5.1 Basic Observations

We begin with the relationship between $\langle\mathsf{CBA}, \mathsf{CBA}\rangle$-free generation and $\langle\mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation. That the former entails the latter is immediate. That the entailment cannot be reversed follows by Theorem 4.1 and the following result:

Proposition 5.1. *For every set S, the double powerset algebra $\mathfrak{P}\mathcal{P}(S)$ is $\langle\mathsf{csub}(\mathfrak{P}\mathcal{P}(S)), \mathsf{CBA}\rangle$-freely generated by the set of canonical generators Γ_S.*

Proof. Consider any complete subalgebra $\mathfrak{B}$ of $\mathfrak{A} = \mathfrak{P}\mathcal{P}(S)$, and function $f : \Gamma_S \to B$. Let $\hat{f} : A \to A$ such that for all $x \in A$:

$$\hat{f}(x) = \{w \subseteq S : \{s \in S : w \in f(s^*)\} \in x\}$$

It is straightforward to show that $\hat{f}$ is a complete homomorphism from $\mathfrak{A}$ to $\mathfrak{A}$ which extends f.

Recall that Γ_S completely generates $\mathfrak{A}$. By inductions on the generation of $\mathfrak{A}$ (as in the proof of Lemma 4.5), it follows first that $\mathrm{im}(\hat{f}) \subseteq B$ whence $\hat{f}$ is a complete homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and second that $\hat{f}$ is the only such homomorphism extending f. $\square$

It is immediate that $\langle\mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation entails $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-free generation. That $\langle\mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation entails complete independent generation is shown in the next result:

Proposition 5.2. *If E $\langle\mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$, then E completely independently generates $\mathfrak{A}$.*

Proof. We first show that E is completely independent. This can in fact be established from the weaker assumption that E $\langle\mathsf{T}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$, for some class T of complete Boolean algebras containing a non-trivial algebra $\mathfrak{B}$. Consider any $D \lhd E$. Let $f : E \to B$ be the function such that $f(e) = \top$ if $e \in D$, and $f(e) = \bot$ otherwise. By $\langle\mathsf{T}, \mathsf{CBA}\rangle$-free generation, f extends to a complete homomorphism $\hat{f}$. Then $\hat{f}(\bigwedge D) = \top$, whence $\bigwedge D > \bot$.

To show that E completely generates $\mathfrak{A}$, let $\mathfrak{B}$ the smallest subalgebra of $\mathfrak{A}$ including E. We show that $\mathfrak{B} = \mathfrak{A}$. Let $\imath$ be the identity map on E. By $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation, $\imath$ extends to a complete homomorphism $\hat{\imath}$ from $\mathfrak{A}$ to $\mathfrak{B}$. $\hat{\imath}$ is also a complete homomorphism from $\mathfrak{A}$ to $\mathfrak{A}$ extending $\imath$, so by $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation, $\hat{\imath}$ is the only such homomorphism. But the identity map on A is such a homomorphism, so $\hat{\imath}$ is the identity map on A. So $\mathfrak{B} = \mathfrak{A}$. □

It follows immediately from Theorems 4.8 and 4.4 that complete independent generation entails $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation, but not vice versa. Moreover, there are complete Boolean algebras $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generated by some set, and not completely independently generated by any set:

Proposition 5.3. *Some complete Boolean algebra is $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generated by a set of its elements, while not being completely independently generated by any set of its elements.*

Proof. We can derive the claim from Theorems 4.4 and 4.8 together with the fact that for any cardinality κ, there are complete Boolean algebras all of whose completely generating sets are of cardinality $\geq \kappa$. For example, the completion of the free Boolean algebra with κ many generators is only completely generated by sets of size $\geq \kappa$; see Koppelberg (1989, p. 211, Ex. 2). Let $\mathfrak{B}$ be the completion of the free Boolean algebra with $\beth_1 = 2^{\aleph_0}$ generators, and $\mathfrak{A} = \mathfrak{B} \times \mathfrak{P}\mathcal{P}(\aleph_0)$. By Theorem 4.4, $\mathfrak{A}$ is $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generated. Assume for contraction that it is completely independently generated by a set E. By Theorem 4.8, E must be at least of size $\beth_1$. But then $\mathfrak{P}\mathcal{P}(\aleph_0)$ must be isomorphic to $\mathfrak{P}\mathcal{P}(S)$ for some set S of cardinality $\beth_1$, which is impossible. □

It only remains to show that complete independent generation does not entail $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-free generation. Since we have seen that $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation entails both complete independent generation and $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-free generation, it immediately follows that complete independent generation does not entail $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation. Moreover, we can show not only that some set which completely independently generates an algebra fails to $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generate it, but more specifically that some algebra which is completely independently generated by a set of its elements is not $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by any set of its elements. However, establishing this relationship between complete independent generation and $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-free generation is more complicated. We therefore devote a separate section to it.

5.2 Complete Independent Generation without $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-Free Generation

For the result of this section, we need a number of further notions for a Boolean algebra $\mathfrak{A}$. First, elements $x, y \in A$ are *disjoint* if $x \wedge y = \bot$. An *antichain* of a $\mathfrak{A}$ is a set $X \subseteq A$ which is pairwise disjoint, i.e., such that any two distinct elements of X are disjoint. $\mathfrak{A}$ satisfies the *countable chain condition* if every antichain of $\mathfrak{A}$ is countable. Finally, $\mathfrak{A}$ is *homogeneous* if for every $x \in A$, the relativization of $\mathfrak{A}$ to x is isomorphic to $\mathfrak{A}$ itself. To these standard notions, we add two non-standard notions which will be helpful as well.

Definition 5.4. *Let $\mathfrak{A}$ be a Boolean algebra and $G \subseteq A$.*

G weakly completely independent *if every $G_0 \subseteq G$ such that $|G \backslash G_0| = |G|$ is completely independent.*

G divisible *if there is some non-zero $x \in A$ such that for all $g \in G$, either $x \leq g$ or $x \leq -g$.*

Theorem 4.8 tells us that to construct an algebra which is completely independently generated, we can start, without loss of generality, with a product $\mathfrak{A} = \mathfrak{B} \times \mathfrak{P}\mathcal{P}(S)$, with S of at least the size of some set of complete generators for $\mathfrak{B}$. We can easily satisfy this cardinality constraint by letting $S = B$. The challenge is now to ensure that $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-free generation fails, for any set $E \subseteq A$. It turns out that this can be done by choosing $\mathfrak{B}$ to be infinite, homogeneous, and satisfying the countable chain condition; such algebras are well-known to exist.

To show that $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-free generation fails in such a case, we establish a number of lemmas. We begin with some relatively basic observations concerning the atomless factor $\mathfrak{B}$. Since we ultimately only care about infinite algebras, we restrict many results to these cases for simplicity. (Note that the only finite homogeneous Boolean algebras are **1** and **2**.)

Lemma 5.5. *Let $\mathfrak{B}$ be an infinite atomless complete Boolean algebra.*

(i) If $\mathfrak{B}$ is completely generated by a set $G \subseteq B$, then $\bigwedge D = \bot$ for every $D \lhd G$.

(ii) If $\mathfrak{B}$ satisfies the countable chain condition, then every weakly completely independent set $G \subseteq B$ is finite.

Proof. Since $\mathfrak{B}$ is atomless, (i) follows by Lemma 4.5. For (ii), assume that $\mathfrak{B}$ has a weakly completely independent set $G \subseteq B$ which is infinite. Then there is an infinite set $G_0 \subseteq G$ such that $|G \backslash G_0| = |G|$. G_0 is completely independent, whence $\{\bigwedge D : D \lhd G_0\}$ is pairwise disjoint, and of cardinality $2^{|G_0|}$, which is uncountable. So $\mathfrak{B}$ does not satisfy the countable chain condition. □

Next, we note that if E $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$, then its complete descriptions conjoin to the atoms of $\mathfrak{A}$:

Lemma 5.6. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be infinite complete Boolean algebras, with $\mathfrak{B}$ being atomless and $\mathfrak{C}$ being atomic. If $\mathfrak{A} = \mathfrak{B} \times \mathfrak{C}$ is $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by a set $E \subseteq A$, then $\{\bigwedge D : D \lhd E\}$ is the set of atoms of $\mathfrak{A}$.*

Proof. To show that $\bigwedge D$ is an atom, for any $D \lhd E$, note first that $\bigwedge D > \bot$: Let $f : E \to \{\bot_{\mathfrak{A}}, \top_{\mathfrak{A}}\}$ be the function such that $f(e) = \top_{\mathfrak{A}}$ iff $e \in D$. By $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-free generation, f extends to $\hat{f}$. Consider any $d \in D$: If $d \in E$, then $\hat{f}(d) = \top_{\mathfrak{A}}$. If $d \notin E$, then $-d \in E$ and $-d \notin D$; so $f(-d) = \bot_{\mathfrak{A}}$ whence $\hat{f}(d) = \top_{\mathfrak{A}}$. So $\hat{f}(\bigwedge D) = \top_{\mathfrak{A}}$, whence $\bigwedge D > \bot_{\mathfrak{A}}$.

To complete the argument for $\bigwedge D$ being an atom, note that $\pi_1(\bigwedge D) = \bot_{\mathfrak{B}}$: $\pi_1(\bigwedge D)$ is $\bigwedge\{\pi_1(d) : d \in D\}$, and since $\{\pi_1(d) : d \in D\} \lhd G$ and G completely generates $\mathfrak{B}$, it follows with Lemma 5.5 (i) that $\pi_1(\bigwedge D) = \bot_{\mathfrak{B}}$. So $\bigwedge D = \langle\bot_{\mathfrak{B}}, c\rangle$ for some $\bot_{\mathfrak{C}} < c \in C$. Thus, if $\bigwedge D$ is not an atom, there are distinct atoms a_0, a_1 of $\mathfrak{A}$ such that $a_1, a_2 < \bigwedge D$. For each atom, there is a corresponding complete homomorphism from $\mathfrak{A}$ to $\{\bot_{\mathfrak{A}}, \top_{\mathfrak{A}}\}$; they both map $\bigwedge D$ to $\top_{\mathfrak{A}}$, and so agree on E, contradicting $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-free generation.

Conversely, for every atom a of $\mathfrak{A}$, there is a corresponding complete homomorphism $\mathfrak{A}$ to $\{\bot_{\mathfrak{A}}, \top_{\mathfrak{A}}\}$, and so some $D \lhd E$ such that $a \leq \bigwedge D$. As just shown, $\bigwedge D$ is an atom, so a must be $\bigwedge D$. Thus every atom is $\bigwedge D$, for some $D \lhd E$. So $\{\bigwedge D : D \lhd E\}$ is the set of atoms of $\mathfrak{A}$. □

The next lemma is central; it shows that in the envisaged algebra $\mathfrak{A}$, the first coordinates of any $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generating set must be divisible:

Lemma 5.7. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be infinite complete Boolean algebras, with $\mathfrak{B}$ being atomless and satisfying the countable chain condition, and $\mathfrak{C}$ being atomic. If $\mathfrak{A} = \mathfrak{B} \times \mathfrak{C}$ is $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by a set $E \subseteq A$, then $\{\pi_1(e) : e \in E\}$ is divisible.*

Proof. Assume for contradiction that $\mathfrak{A}$ is $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by E and $G = \{\pi_1(e) : e \in E\}$ is not divisible. We derive of a contradiction from this assumption in four steps.

In the FIRST STEP, we identify a particular function $f : E \to A$. Every finite subset of a Boolean algebra is divisible, so G is infinite. It follows by Lemma 5.5 (ii) that G is not weakly completely independent. So there is some $G_0 \subseteq G$ which is not completely independent such that $|G \backslash G_0| = |G|$. As G_0 is not completely independent, there is some $D_0 \lhd G_0$ such that $\bigwedge D_0 = \bot_{\mathfrak{B}}$. Let f_0 be a function from G to B such that $f_0(g) = \top_{\mathfrak{B}}$ if $g \in G_0 \cap D_0$, $f_0(g) = \bot_{\mathfrak{B}}$ if $g \in G_0 \backslash D_0$, and $G \subseteq \mathrm{im}(f_0)$. Such a function exists for cardinality reasons. Define $f : E \to A$ as $\langle b, c\rangle \mapsto \langle f_0(b), c\rangle$. By $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-free generation, there is a complete homomorphism $\hat{f}$ from $\mathfrak{A}$ to $\mathfrak{A}$ extending f.

In the SECOND STEP, we show that for every $D \lhd E$, $\hat{f}(\bigwedge D) = \bigwedge D$. So consider any $D \lhd E$. We first describe $\bigwedge D$: As shown in Lemma 5.6, $\bigwedge D$ is an atom, whence $\bigwedge D = \langle \bot_{\mathfrak{B}}, \pi_2 \bigwedge D\rangle$. It thus suffices to show that the two coordinates of $\hat{f}(\bigwedge D)$ are $\bot_{\mathfrak{B}}$ and $\pi_2(\bigwedge D)$. We consider them in turn.

First coordinate: $G \subseteq \mathrm{im}(f_0)$, so for every $g \in G$, there is an $e \in E$ such that $\pi_1 \hat{f}(e) = g$. If $e \in D$, then $\pi_1 \bigwedge \hat{f}(D) \leq g$. If $e \notin D$, then $-e \in D$; so $\pi_1 \hat{f}(-e) = -g$, whence $\pi_1 \hat{f}(\bigwedge D) \leq -g$. It follows that there is an $X \lhd G$ such that $\pi_1 \hat{f}(\bigwedge D) \leq \bigwedge X$. By Lemma 5.5 (i), $\bigwedge X = \bot_{\mathfrak{B}}$, so $\pi_1 \hat{f}(\bigwedge D) = \bot_{\mathfrak{B}}$.

Second coordinate: Consider any $\langle b, c\rangle \in D$. If $\langle b, c\rangle \in E$, then $\pi_2 \hat{f}(\langle b, c\rangle) = c$. If $\langle b, c\rangle \notin E$, then $\langle -b, -c\rangle \in E$. So $\hat{f}(\langle b, c\rangle) = -\hat{f}(\langle -b, -c\rangle) = \langle f_0(-b), c\rangle$. So $\pi_2 \hat{f}(\langle b, c\rangle) = c$. So for all $d \in D$, $\pi_2 \hat{f}(d) = \pi_2(d)$, whence $\pi_2 \hat{f}(\bigwedge D) = \pi_2(\bigwedge D)$.

In the THIRD STEP, we show that for all $g \in G$, $\hat{f}(\langle g, \bot_{\mathfrak{C}}\rangle) = \langle f_0(g), \bot_{\mathfrak{C}}\rangle$ and $\hat{f}(\langle -g, \bot_{\mathfrak{C}}\rangle) = \langle -f_0(g), \bot_{\mathfrak{C}}\rangle$. Since $\langle \bot_{\mathfrak{B}}, \top_{\mathfrak{C}}\rangle$ is the disjunction of atoms of $\mathfrak{A}$, Lemma 5.6 entails that $\langle \bot_{\mathfrak{B}}, \top_{\mathfrak{C}}\rangle = \bigvee\{\bigwedge D : D \lhd E\}$. With the claim established in the second step, it follows that $\hat{f}$ maps $\langle \bot_{\mathfrak{B}}, \top_{\mathfrak{C}}\rangle$ to itself; consequently, it also maps $\langle \top_{\mathfrak{B}}, \bot_{\mathfrak{C}}\rangle$ to itself.

Consider first any $\langle b, c\rangle \in E$. Since $\langle b, \bot_{\mathfrak{C}}\rangle = \langle b, c\rangle \wedge \langle \top_{\mathfrak{B}}, \bot_{\mathfrak{C}}\rangle$, it follows:

$$\hat{f}(\langle b, \bot_{\mathfrak{C}}\rangle) = \hat{f}(\langle b, c\rangle) \wedge \hat{f}(\langle \top_{\mathfrak{B}}, \bot_{\mathfrak{C}}\rangle) = \langle f_0(b), c\rangle \wedge \langle \top_{\mathfrak{B}}, \bot_{\mathfrak{C}}\rangle = \langle f_0(b), \bot_{\mathfrak{C}}\rangle$$

Similarly:

$$\hat{f}(\langle b, \top_{\mathfrak{C}}\rangle) = \hat{f}(\langle b, c\rangle) \vee \hat{f}(\langle \bot_{\mathfrak{B}}, \top_{\mathfrak{C}}\rangle) = \langle f_0(b), c\rangle \vee \langle \bot_{\mathfrak{B}}, \top_{\mathfrak{C}}\rangle = \langle f_0(b), \top_{\mathfrak{C}}\rangle$$

Thus $\hat{f}(\langle -b, \bot_{\mathfrak{C}}\rangle) = -\hat{f}(\langle b, \top_{\mathfrak{C}}\rangle) = -\langle f_0(b), \top_{\mathfrak{C}}\rangle = \langle -f_0(b), \bot_{\mathfrak{C}}\rangle$.

In the final and FOURTH STEP, the assumption that $\hat{f}$ is a complete homomorphism will be contradicted. Since $\bigwedge D_0 = \bot_{\mathfrak{B}}$, $\bigwedge\{\langle d, \bot_{\mathfrak{C}}\rangle : d \in D_0\} = \bot_{\mathfrak{A}}$. However, using the observations established in the third step:

$$\hat{f}(\bigwedge\{\langle d, \bot_{\mathfrak{C}}\rangle : d \in D_0\}) = (\bigwedge\{\hat{f}\langle d, \bot_{\mathfrak{C}}\rangle) : d \in D_0\}) = \langle \top_{\mathfrak{B}}, \bot_{\mathfrak{C}}\rangle$$

So $\hat{f}(\bot_{\mathfrak{A}}) = \langle \top_{\mathfrak{B}}, \bot_{\mathfrak{C}}\rangle$, which contradicts $\hat{f}$ being a homomorphism. □

With this lemma, we can show that the envisaged algebra $\mathfrak{A}$ indeed fails to be $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generated:

Proposition 5.8. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be infinite complete Boolean algebras, with $\mathfrak{B}$ being homogeneous and satisfying the countable chain condition, and $\mathfrak{C}$ being atomic. Then $\mathfrak{A} = \mathfrak{B} \times \mathfrak{C}$ is not $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by any set.*

Proof. Assume for contradiction that there is a set $E \subseteq A$ which $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$. Since $\mathfrak{B}$ is infinite and homogeneous, it is atomless. By Lemma 5.7, $G = \{\pi_1(e) : e \in E\}$ is divisible. Let $y \in B$ be a witness of this. As $\mathfrak{B}$ is atomless, there is a non-zero $x < y$. Let $x' = y \wedge -x$. Homogeneity is preserved under relativization, so $\mathfrak{B}$ relativized to y is infinite and homogeneous.

By homogeneity, this relativized algebra has an automorphism f mapping x to x'; see Koppelberg (1989, p. 135, Prop. 9.13). Define a function g on B such that for all $z \in B$:

$$g(z) = f(z \wedge y) \vee (z \wedge -y)$$

It is routine to confirm that g is an automorphism of $\mathfrak{B}$. (One way to see this is to note that if f and f' are automorphisms of $\mathfrak{B}$ relativized to y and $-y$, respectively, then $z \mapsto f(z \wedge y) \vee f'(z \wedge -y)$ is an automorphism of $\mathfrak{B}$.) Similarly, it is routine to derive that g maps every element of G to itself from the fact that y witnesses the divisibility of G.

Finally, let ι be the function mapping every element of E to itself, and ι_1 the function mapping every element of A to itself. ι_1 is obviously an automorphism of $\mathfrak{A}$ extending ι. Let ι_2 be the function mapping every $\langle b, c\rangle \in A$ to:

$$\iota_2(\langle b, c\rangle) = \langle g(b), c\rangle$$

Using the fact that g is an automorphism of $\mathfrak{B}$, it is routine to show that ι_2 is an automorphism of $\mathfrak{A}$. Further, since g maps every element of G to itself, ι_2 maps every element of E to itself. So, both ι_1 and ι_2 are automorphisms – and so complete endomorphisms – of $\mathfrak{A}$, and both extend ι. But since g does not map every element of B to itself, $\iota_2 \neq \iota_1$. This contradicts the uniqueness of $\hat{\iota}$ required by $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-free generation. □

Theorem 5.9. *Some complete Boolean algebra is completely independently generated by a set of elements, while not being $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by any set of elements.*

Proof. For any infinite set I, consider the partial order of finite partial functions from I to 2, ordered by reverse subset inclusion. The corresponding partial order topology determines a complete Boolean algebra $\mathfrak{B}$. This algebra is infinite and homogeneous, and so atomless, and satisfies the countable chain condition. The construction is well-known, so we omit the details; see Koppelberg (1989, p. 181 & p. 64, Ex. 6) and Bell (2005, p. 50). By Theorem 4.8, $\mathfrak{A} = \mathfrak{B} \times \mathfrak{P}\mathcal{P}(B)$ is completely independently generated by a set of elements. But by Proposition 5.8, $\mathfrak{A}$ is not $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by any set of elements. □

5.3 Open Questions and Further Results

We have now established all the relationships recorded in Figure 1. A number of questions remain open.

First, we have seen that any double powerset algebra $\mathfrak{A}$ is $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-freely generated by its canonical generators. We do not know whether these are the only examples, up to isomorphism:

Open Question 5.10. *Is every complete Boolean algebra $\mathfrak{A}$ which is $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-freely generated by a set of its elements isomorphic to a double powerset algebra?*

Second, as noted above, it is immediate that $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation entails $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-free generation. We do not know whether the converse is the case as well:

Open Question 5.11. *Is every complete Boolean algebra $\mathfrak{A}$ which is $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by a set of its elements also $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-freely generated by this set?*

What we do know is that this question has a negative answer if there are complete Boolean algebras with only one complete endomorphism. Of course, every algebra has at least one complete endomorphism, the function mapping every element to itself. Call a function *non-trivial* if it does not map every element on which it is defined to itself. We can show that an algebra $\mathfrak{A}$ without non-trivial complete endomorphisms is $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by a set of its elements but not $\langle\mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-freely generated by any set of its elements. Indeed, we can show that such an algebra is not completely independently generated by any set of its elements; since $\langle\mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-free generation entails complete independent generation but not vice versa, this is a strictly stronger condition.

Proposition 5.12. *Any infinite complete Boolean algebra $\mathfrak{A}$ without non-trivial complete endomorphisms has a set of elements which $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generates it, but no set of elements which completely independently generates it.*

Proof. Let $\mathfrak{A}$ be a complete Boolean algebra without non-trivial complete endomorphisms. $\emptyset$ $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$: There is a single function $f : \emptyset \to \mathfrak{B}$, which extends to a unique complete endomorphism extending f, namely the identity function.

Assume for contradiction that $\mathfrak{A}$ is completely independently generated by some set $E \subseteq A$. Since A is infinite, E must be infinite as well. By Theorem 4.8, $\mathfrak{A}$ is isomorphic to the product of an atomless complete Boolean algebra with $\mathfrak{P}\mathcal{P}(E)$. Thus $\mathfrak{A}$ contains distinct atoms a, a'. However, for any two atoms of a powerset algebra, there is an automorphism f which interchanges them. Thus, $\mathfrak{A}$ has a non-trivial automorphism $\langle b, c\rangle \to \langle b, f(c)\rangle$, which is a complete endomorphism. This contradicts the assumption that $\mathfrak{A}$ has no non-trivial complete endomorphism. □

To our knowledge, the question whether there are complete Boolean algebras without non-trivial complete endomorphisms has not be considered in the literature. However, various related conditions have been considered, and shown to be instantiated. For example, a Boolean algebra is called *rigid* if it has no non-trivial automorphisms. Any Boolean algebra with more than two elements has non-trivial endomorphisms, but there are certain Boolean algebras which have very few endomorphisms; these are called *endo-rigid*. (We omit the definition of endo-rigidity, which is complicated to state.) Endo-rigid Boolean algebras are rigid. Various complete Boolean algebras have been shown to be rigid, and certain Boolean algebras have been shown to be endo-rigid; see Bekkali and Bonnet (1989) and Monk (1989). These results give us some hope that our related question has a positive answer as well:

Open Question 5.13. *Is there a complete Booolean algebra without non-trivial complete endomorphisms?*

Finally, we have seen in Theorem 5.9 that there are algebras $\mathfrak{A}$ completely independently generated by some set of elements but not $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by any set of elements. More generally, Proposition 5.8 describes a class of algebras $\mathfrak{A}$ not $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by any set of elements. A simpler argument, in some ways similar, gives us many more examples of algebras $\mathfrak{A}$ not $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by any set of elements:

Proposition 5.14. *No infinite homogeneous complete Boolean algebra $\mathfrak{A}$ is $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generated by any set of elements.*

Proof. Consider any infinite homogeneous complete Boolean algebra $\mathfrak{A}$ and $E \subseteq A$. Assume for contradiction that E $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$. Let ι be the function mapping every element of E to itself, and ι_1 the function mapping every element of A to itself. ι_1 is an automorphism extending ι. It suffices to produce a second automorphism ι_2 which extends ι: automorphisms are complete endomorphisms, so the distinctness of ι_1 and ι_2 contradicts the uniqueness of $\hat{\iota}$ required for E to $\langle\mathfrak{A}, \mathsf{CBA}\rangle$-freely generate $\mathfrak{A}$. We distinguish two cases:

Assume first that $\bigwedge E = \top$. Then $E \subseteq \{\top\}$. Since $\mathfrak{A}$ is infinite and homogeneous, it has a non-trivial automorphism ι_2. Any automorphism maps $\top$ to itself, so ι_2 extends ι, as required.

Assume now that $\bigwedge E < \top$. As noted in the proof of Proposition 5.2, it follows from the assumption that E $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely generates $\mathfrak{A}$ that E is completely independent. Consequently, $\bot < \bigwedge E$. Since $\mathfrak{A}$ is homogeneous, it is atomless. So, there is a non-zero $e < \bigwedge E$. Let $b = \bigwedge E \wedge -e$. By homogeneity, there is an automorphism f of $\mathfrak{A}$ restricted to $\bigwedge E$ mapping e to b. f extends to an automorphism ι_2 of $\mathfrak{A}$, letting $\iota_2(x) = f(x \wedge \bigwedge E) \vee (x \wedge -\bigwedge E)$ for all $x \in A$. ι_2 is non-trivial, but maps every element of E to itself, as required. $\square$

6 Conclusion

We began with LA, the principle that every proposition is a truth-functional combination of elementary propositions. We noted that in the setting of arbitrary Boolean algebras (rather than the setting of *complete* Boolean algebras), the natural algebraic ways of formalizing this principle all lead to the same notion, of an independently (or, equivalently, freely) generated Boolean algebra $\mathfrak{A}$. In the finite case, this gives us exactly the double powerset algebras, and so matches how a number of authors have modeled logical atomism in Boolean algebras. However, in the infinite case, free generation comes apart starkly from the double powerset picture. One question we noted above was whether harmony can be restored in the infinite domain once we move to complete Boolean algebras, and include arbitrary conjunctions (greatest lower bounds) in the various algebraic regimentations of LA.

To some extent, we have seen that this hope for a rapprochement is vindicated: For any set S, the canonical generators Γ_S $\langle \mathsf{csub}(\mathfrak{A}), \mathsf{CBA}\rangle$-freely generate the double powerset algebra $\mathfrak{A} := \mathfrak{P}\mathcal{P}(S)$, and it follows from this that they also complete independently, $\langle \mathfrak{A}, \mathsf{CBA}\rangle$-freely, and $\langle \mathbf{2}, \mathsf{CBA}\rangle$-freely generate this algebra. However, we have also seen some ways in which these conditions don't exactly vindicate the double powerset picture. First, the remaining condition of $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-free generation does not fall in line, as Γ_S does not $\langle \mathsf{CBA}, \mathsf{CBA}\rangle$-freely generate $\mathfrak{P}\mathcal{P}(S)$ when S is infinite. Second, there are cases of both complete independent generation and $\langle \mathbf{2}, \mathsf{CBA}\rangle$-free generation in which the algebra is not a double powerset algebra.

Formal regimentation in Boolean algebras thus reveals that the informal idea captured in LA can be made precise in a number of different ways which lead to substantially different results. This applies to the various options in the context of Boolean algebras considered here. But it also applies to various ways of capturing LA which do not appeal to Boolean algebras. As noted above, our own preferred way or regimenting talk of propositions is of this kind, namely a regimentation in higher-order logic. We hope to consider this in more detail in future work.

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